# Political Science 209 - Fall 2018

Probability II

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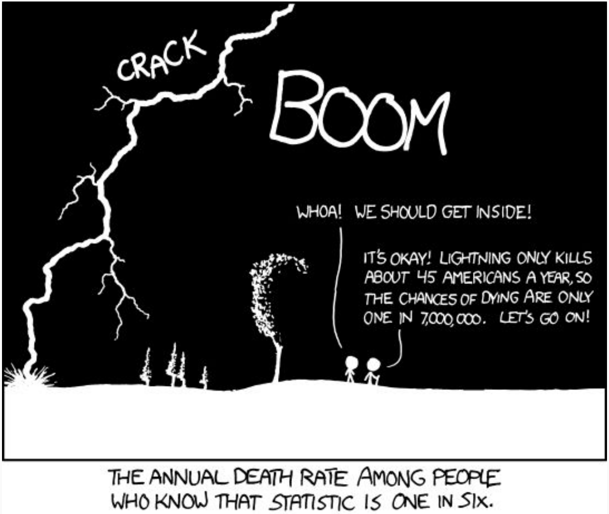
# Conditional Probability



# Conditional Probability

Sometimes information about one event can help inform us about likelihood of another event

Examples?

# Conditional Probability

Sometimes information about one event can help inform us about likelihood of another event

Examples?

- What is the probability of rolling a 5 and then a 6?
- What is the probability of rolling a 5 and then a 6 given that we rolled a 5 first?

# Conditional Probability

If it is cloudy outside, gives us additional information about likelihood of rain

If we know that one party will win the House, makes it more likely that party will win certain Senate races

# Independence

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If the occurrence of one event (A) gives us information about likelihood of another event, then the two events are not independent.

Independence of two events implies that information about one event does not help us in knowing whether the second event will occur.

# Independence

For many real world examples, independence does not hold

Knowledge about other events allows us to improve guesses/probability calculations

# Independence

When two events are independence, the probability of both happening is equal to the individual probabilities multiplied together

# Conditional Probability

$P(A \mid B)$

*Probability of A given/conditional that B has happened*

# Conditional Probability

$\mathrm{P(A \mid B)} = \frac{P(AandB)}{P(B)}$

*Probability of A and B happening (joint) divided by probability of B happening (marginal)*

# Conditional Probability

Definitions:

P(A and B) - joint probability

P(A) - marginal probability

# Conditional Probability

P(rolled 5 then 6) = ?

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P(rolled 5 then 6) $= \frac{1}{36}$

P(rolled 5 then 6 | 5 first) $= \frac{P(5 then 6)}{P(5)}$

# Conditional Probability

P(rolled 5 then 6) $=$ ?

P(rolled 5 then 6) $= \frac{1}{36}$

P(rolled 5 then 6 | 5 first) $= \frac{P(5 then 6)}{P(5)}$

$\frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$

The probability that it is Friday and that a student is absent is 0.03. What is the probability that student is absent, given that it is Friday?

$P(\text{absent} \mid \text{Friday}) = ?$

# Conditional Probability

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$P(\text{absent} \mid \text{Friday}) = ?$

$P(\text{absent} \mid \text{Friday}) = \frac{0.03}{0.2} = 0.15$

# Conditional Probability

P(A | B) = $\frac{P(AandB)}{P(B)}$

Also means:

P(A and B) = P(A | B) P(B)

# Independence

If A and B are independent, then

- $P(A \mid B) = P(A)$ & $P(B \mid A) = P(B)$
- $P(A \text{ and } B) = P(A) \times P(B)$

# Independence

If A|C and B|C are independent, then

- P(A and B | C) = P(A |C) $\times$ P(B | C)

# Probability Problems

What is the probability of drawing any card between 2 and 10, or jack, queen, king in any color?

What is the probability of drawing two kings from a full deck of cards?

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$\text{P}(2 \text{ kings}) = \frac{4}{52} \times ?$

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$P(2 \text{ kings}) = \frac{4}{52} \times ?$

$P(2 \text{ kings}) = \frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$

# Probability Problems

| Annual income | Took 209 | Took 309 | TOTAL |
|---|---|---|---|
| Under \$50,000 | 36 | 24 | 60 |
| \$50,000 to \$100,000 | 109 | 56 | 165 |
| over \$100,000 | 35 | 40 | 75 |
| Total | 180 | 120 | 300 |

Is the probability of making over \$100,000 and the probability of having taken 309 independent?

# Probability Problems

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Is the probability of making over \$100,000 and the probability of having taken 309 independent?

P(over \$100k & 309) = P(over \$100k) $\times$ P(309)?

# Probability Problems

| Annual income | Took 209 | Took 309 | TOTAL |
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| Total | 180 | 120 | 300 |

What is the probability of any student making over \$100,000?

| Annual income | Took 209 | Took 309 | TOTAL |
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What is the probability of a student making over \$100,000, conditional that he took 309?

# Probability Problems

| Annual income | Took 209 | Took 309 | TOTAL |
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| $50,000 to $100,000 | 109 | 56 | 165 |
| over $100,000 | 35 | 40 | 75 |
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What is the probability of a having taken 309, conditional on making over $100,000?

# The Monty Hall Paradox!

What is the Monty Hall Paradox?

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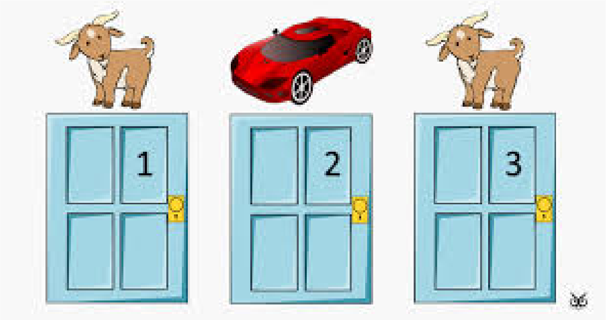

# The Monty Hall Paradox!

What is the probability of winning a car when not switching?

$P(\text{car}) = ?$

What is the probability of winning a car when not switching?

$P(\text{car}) = \frac{1}{3}$

What is the probability of winning a car when switching?

# The Monty Hall Paradox!

What is the probability of winning a car when switching?

Consider two scenarios: picking door with car first and picking door with goat first

# The Monty Hall Paradox: switching

Consider two scenarios: picking door with car first and picking door with goat first

1. What is the probability of getting the car when switching after picking the car first?
2. What is the probability of getting the car when switching after picking a goat first?

# The Monty Hall Paradox: switching

P(car when switching) = P(car | car first)$\times$ P(car first) + P(car | goat first) $\times$ P(goat first)

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P(car when switching) = $0 \times \frac{1}{3} + 1 \times \frac{2}{3}$

# The Monty Hall Paradox: switching

P(car when switching) $=$ P(car $|$ car first)$\times$ P(car first) $+$ P(car $|$ goat first) $\times$ P(goat first)

P(car when switching) $= 0 \times \frac{1}{3} + 1 \times \frac{2}{3}$

P(car when switching) $= \frac{2}{3}$

# The Monty Hall Paradox: in *R*

```
sims <- 1000
doors <- c("goat", "goat", "car")
result.switch <- result.noswitch <- rep(NA, sims)
for (i in 1:sims) {
    ## randomly choose the initial door
first <- sample(1:3, size = 1)
result.noswitch[i] <- doors[first]
remain <- doors[-first] # remaining two doors
## Monty chooses one door with a goat
monty <- sample((1:2)[remain == "goat"], size = 1)
result.switch[i] <- remain[-monty]
}
mean(result.noswitch == "car")
mean(result.switch == "car")
```

# Bayes' Rule/Theorem

How should we update our beliefs about event A after learning about some data related to the event?

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Example: What is the probability of a person developing lung cancer?

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Example: What is the probability of a person developing lung cancer?

How does the probability change once we learn about the person's smoking habits?

# Bayes Rule/Theorem

$P(A \mid B) = \frac{P(B|A)P(A)}{P(B)}$

P(A) : prior probability of event A

P(A | B): posterior probability of event A given observed data B

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P(B | A): probability of observing B given A

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P(B | A) $\times$ P(A) ?

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P(B | A) $\times$ P(A) = P(B and A)

# Bayes Rule/Theorem

$$\mathrm{P(A \mid B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(BandA)}{P(B)}$$

# Bayes Rule/Theorem

$$\mathrm{P(A \mid B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)}$$

# Bayes Rule/Theorem

- Does your doctor know Bayes' rule? Cause he/she should!
- Example of medical tests:
  - every test comes with a reliability/accuracy
  - remember: false positive, false negative, etc

# Bayes Rule/Theorem

What is the probability of being pregnant, given that you have a positive test?

$\mathrm{P}(\mathrm{p} \mid + ) = \frac{P(+|p)P(p)}{P(+)}$

# Bayes Rule/Theorem

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P(p | + ) = $\frac{P(+|p)P(p)}{P(+)}$

Decompose P(+), say test is 99 % accurate

# Bayes Rule/Theorem

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$$\text{P(preg} \mid + \text{)} = \frac{P(+|p)P(p)}{P(+)} = \frac{P(+|p)P(p)}{P(+|p)P(p)+P(+|\text{not p})P(\text{not p})}$$

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$$P(p \mid +) = \frac{0.99P(p)}{0.99P(p)+0.05P(\text{not p})}$$

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$\text{P(p} \mid +) = \frac{0.99P(p)}{0.99P(p)+0.05P(\text{not p})}$

$\text{P(p} \mid +) = \frac{0.990.5}{0.99\times0.5+0.05\times0.5} = 0.95$

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# Bayes Rule/Theorem

But what happens if your prior probability is stronger?

$\text{P(p} \mid +) = \frac{0.99P(p)}{0.99P(p)+0.05P(\text{not p})}$

$\text{P(p} \mid +) = \frac{0.990.2}{0.99\times0.2+0.05\times0.8} = 0.83$

# Bayes Rule/Theorem

But what happens if your prior probability is stronger?

$$P(p \mid +) = \frac{0.990.05}{0.99 \times 0.05 + 0.01 \times 0.95} = 0.51$$

# Bayes Rule/Theorem

Many of our tests are not this good and disease is very rare:

- high-risk for down syndrome test
- $P(+ \mid DS) = 0.86$
- $P(+ \mid \text{not } DS) = 0.05$
- $P(DS) = 0.003$

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$$\text{P(DS} \mid +) = \frac{0.86 \times 0.003}{0.86 \times 0.003 + 0.05 * 0.997}$$

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$$\text{P(DS} \mid +) = \frac{0.86\times 0.003}{0.86\times 0.003+0.05*0.997} = 0.049$$

Changes with age!